RECENT RESULTS ON THE ANNIHILATOR GRAPH OF A COMMUTATIVE RING: A SURVEY

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ABSTRACT. Let R be a commutative ring with nonzero identity, $Z(R)$ be its set of zero-divisors, and if $a \in Z(R)$, then let $ann_R(a) = \{d \in R \mid da = 0\}$. The annihilator graph of R is the (undirected) graph $AG(R)$ with vertices $Z(R)^* = Z(R) \setminus \{0\}$, and two distinct vertices x and y are adjacent if and only if $ann_R(xy) \neq ann_R(x) \cup ann_R(y)$. It follows that each edge (path) of the zero-divisor graph $\Gamma(R)$ is an edge (path) of $AG(R)$. The extended zero-divisor graph of R is the undirected (simple) graph $EG(R)$ with the vertex set $Z(R)^*$, and two distinct vertices x and y are adjacent if and only if either $Rx \cap ann_R(y) \neq \{0\}$ or $Ry \cap ann_R(x) \neq \{0\}$. Hence it follows that the zero-divisor graph $\Gamma(R)$ is a subgraph of $EG(R)$. In this paper, we collect some properties (many are recent) of the two graphs $AG(R)$ and $EG(R)$.

1. INTRODUCTION

Let R be a commutative ring with nonzero identity, and let $Z(R)$ be its set of zero-divisors. Over the past several years, there has been considerable attention in the literature to associating graphs with commutative rings (and other algebraic structures) and studying the interplay between ring-theoretic and graph-theoretic properties; see the recent survey articles [10] and [47]. For example, as in [16], the *zero-divisor graph* of R is the (simple) graph $\Gamma(R)$ with vertices $Z(R) \setminus \{0\}$, and distinct vertices x and y are adjacent if and only if $xy = 0$. This concept is due to Beck [29], who let all the elements of R be vertices and was mainly interested in colorings. The zero-divisor graph of a ring R has been studied extensively by many authors, for example see([1]-[3], [11], [20]-[21], [39]-[44], [48]-[54], [58]). We recall from [12], the *total graph* of R, denoted by $T(\Gamma(R))$ is the (undirected) graph with all elements of R as vertices, and for distinct $x, y \in R$, the vertices x and y are adjacent if and only if $x+y \in Z(R)$. The total graph (as in [12]) has been investigated in [8], [7], [6], [5], [47], [49], [52], [36] and [56]; and several variants of the total graph have been studied in [4], [13], [14], [15], [19], [28], [35], [32], [33], [34], [37], [38], and [45]. Let $a \in Z(R)$ and let $ann_R(a) = \{r \in R \mid ra = 0\}$. In 2014, Badawi [23] introduced the *annihilator graph of R*. We recall from [23] that the annihilator graph of R is the (undirected) graph $AG(R)$ with vertices $Z(R)^* = Z(R) \setminus \{0\}$, and two distinct vertices x and y are adjacent if and only if $ann_R(xy) \neq ann_R(x) \cup ann_R(y)$. It follows that each edge (path) of the classical zero-divisor of R is an edge (path) of $AG(R)$. For Further investigations of $AG(R)$, see [24], [25], and [31]. The authors in [26] and [27] introduced the *extended zero-divisor graph of R*. We recall from [26] that the extended zero-divisor graph of R is the undirected (simple) graph $EG(R)$

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with the vertex set $Z(R)^*$, and two distinct vertices x and y are adjacent if and only if either $Rx \cap ann_R(y) \neq \{0\}$ or $Ry \cap ann_R(x) \neq \{0\}$. Hence it follows that the zero-divisor graph $\Gamma(R)$ is a subgraph of $EG(R)$.

Let G be a (undirected) graph. We say that G is *connected* if there is a path between any two distinct vertices. For vertices x and y of G, we define $d(x, y)$ to be the length of a shortest path from x to y ($d(x, x) = 0$ and $d(x, y) = \infty$ if there is no path). Then the *diameter* of G is $diam(G) = \sup\{$ d$(x, y) \mid x$ and y are vertices of $G\}$. The *girth* of G, denoted by $gr(G)$, is the length of a shortest cycle in G ($gr(G) = \infty$ if G contains no cycles).

A graph G is *complete* if any two distinct vertices are adjacent. The complete graph with n vertices will be denoted by K^n (we allow n to be an infinite cardinal). A *complete bipartite graph* is a graph G which may be partitioned into two disjoint nonempty vertex sets A and B such that two distinct vertices are adjacent if and only if they are in distinct vertex sets. If one of the vertex sets is a singleton, then we call G a *star graph*. We denote the complete bipartite graph by $K^{m,n}$, where $|A| = m$ and $|B| = n$ (again, we allow m and n to be infinite cardinals); so a star graph is a $K^{1,n}$ and $K^{1,\infty}$ denotes a star graph with infinitely many vertices. By $\overline{G}$, we mean the complement graph of G. Let G_1, G_2 be two graphs. The join of G_1 and G_2, denoted by $G_1 \vee G_2$, is a graph with the vertex set $V(G_1 \cup G_2) = V(G_1) \cup V(G2)$ and edge set $E(G_1 \cup G_2) = E(G_1) \cup E(G_2) \cup \{u - v | u \in G_1, v \in G_2\}$. Finally, let $\overline{K}^{m,3}$ be the graph formed by joining $G_1 = K^{m,3}$ ($= A \cup B$ with $|A| = m$ and $|B| = 3$) to the star graph $G_2 = K^{1,m}$ by identifying the center of G_2 and a point of B.

Throughout, R will be a commutative ring with nonzero identity, $Z(R)$ its set of zero-divisors, $Nil(R)$ its set of nilpotent elements, $U(R)$ its group of units, $T(R)$ its total quotient ring, and $Min(R)$ its set of minimal prime ideals. For any $A \subseteq R$, let $A^* = A \setminus \{0\}$. We say that R is *reduced* if $Nil(R) = \{0\}$ and that R is *quasi-local* if R has a unique maximal ideal. A prime ideal P of R is called an associated prime ideal, if $ann_R(x) = P$, for some non-zero element $x \in R$. The set of all associated prime ideals of R is denoted by $Ass(R)$, and $\sum = \{ann_R(x) | 0 \neq x \in R\}$. The distance between two distinct vertices a, b of $\Gamma(R)$ is denoted by $d_{\Gamma(R)}(a, b)$. If $AG(R)$ is identical to $\Gamma(R)$, then we write $AG(R) = \Gamma(R)$; otherwise, we write $AG(R) \neq \Gamma(R)$. As usual, $\mathbb{Z}$ and $\mathbb{Z}_n$ will denote the integers and integers modulo n, respectively.

2. Basic properties of $AG(R)$

We recall the following basic results from [23].

Theorem 2.1. *([23, Theorem 2.2]) Let R be a commutative ring with $|Z(R)^*| \geq 2$. Then $AG(R)$ is connected and $diam(AG(R)) \leq 2$.*

Theorem 2.2. *([23, Theorem 2.4]) Let R be a commutative ring. Suppose that $x - y$ is an edge of $AG(R)$ that is not an edge of $\Gamma(R)$ for some distinct $x, y \in Z(R)^*$. If $xy^2 \neq 0$ and $x^2 y \neq 0$, then there is a $w \in Z(R)^*$ such that $x - w - y$ is a path in $AG(R)$ that is not a path in $\Gamma(R)$, and hence $C : x - w - y - x$ is a cycle in $AG(R)$ of length three and each edge of C is not an edge of $\Gamma(R)$.*

In view of Theorem 2.2, we have the following result.

Corollary 2.3. *([23, Corollary 2.5])*

Let R be a reduced commutative ring. Suppose that $x - y$ is an edge of $AG(R)$ that is not an edge of $\Gamma(R)$ for some distinct $x, y \in Z(R)^*$. Then there is a $w \in ann_R(xy) \setminus \{x, y\}$ such that $x - w - y$ is a path in $AG(R)$ that is not a path in $\Gamma(R)$, and hence $C : x - w - y - x$ is a cycle in $AG(R)$ of length three and each edge of C is not an edge of $\Gamma(R)$.

In light of Corollary 2.3, the following result follows.

Theorem 2.4. *([23, Theorem 2.6]) Let R be a reduced commutative ring and suppose that $AG(R) \neq \Gamma(R)$. Then $gr(AG(R)) = 3$. Furthermore, there is a cycle C of length three in $AG(R)$ such that each edge of C is not an edge of $\Gamma(R)$.*

In view of Theorem 2.2, the following is an example of a non-reduced commutative ring R where $x - y$ is an edge of $AG(R)$ that is not an edge of $\Gamma(R)$ for some distinct $x, y \in Z(R)^*$, but every path in $AG(R)$ of length two from x to y is also a path in $\Gamma(R)$.

Example 2.5. *Let $R = \mathbb{Z}_8$. Then $2 - 6$ is an edge of $AG(R)$ that is not an edge of $\Gamma(R)$. Now $2 - 4 - 6$ is the only path in $AG(R)$ of length two from 2 to 6 and it is also a path in $\Gamma(R)$. Note that $AG(R) = K^3$, $\Gamma(R) = K^{1,2}$, $gr(\Gamma(R)) = \infty$, $gr(AG(R)) = 3$, $diam(\Gamma(R)) = 2$, and $diam(AG(R)) = 1$.*

The following is an example of a non-reduced commutative ring R such that $AG(R) \neq \Gamma(R)$ and if $x - y$ is an edge of $AG(R)$ that is not an edge of $\Gamma(R)$ for some distinct $x, y \in Z(R)^*$, then there is no path in $AG(R)$ of length two from x to y.

Example 2.6.
1. *Let $R = \mathbb{Z}_2 \times \mathbb{Z}_4$ and let $a = (0,1), b = (1,2)$, and $c = (0,3)$. Then $a - b$ and $c - b$ are the only two edges of $AG(R)$ that are not edges of $\Gamma(R)$, but there is no path in $AG(R)$ of length two from a to b and there is no path in $AG(R)$ of length two from c to b. Note that $AG(R) = K^{2,3}$, $\Gamma(R) = \overline{K}^{1,3}$, $gr(AG(R)) = 4$, $gr(\Gamma(R)) = \infty$, $diam(AG(R) = 2$, and $diam(\Gamma(R)) = 3$.*
2. *Let $R = \mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^2)$. Let $x = X + (X^2) \in \mathbb{Z}_2[X]/(X^2)$, $a = (0,1), b = (1, x)$, and $c = (0, 1 + x)$. Then $a - b$ and $c - b$ are the only two edges of $AG(R)$ that are not edges of $\Gamma(R)$, but there is no path in $AG(R)$ of length two from a to b and there is no path in $AG(R)$ of length two from c to b. Again, note that $AG(R) = K^{2,3}$, $\Gamma(R) = \overline{K}^{1,3}$, $gr(AG(R)) = 4$, $gr(\Gamma(R)) = \infty$, $diam(AG(R) = 2$, and $diam(\Gamma(R)) = 3$.*

If $AG(R) \neq \Gamma(R)$ and $gr(AG(R)) = 4$, then the following result characterize, up to isomorphism, all such rings.

Theorem 2.7. *([23, Theorem 2.9]) Let R be a commutative ring and suppose that $AG(R) \neq \Gamma(R)$. Then the following statements are equivalent:*
1. *$gr(AG(R)) = 4$;*
2. *$gr(AG(R)) \neq 3$;*
3. *If $x - y$ is an edge of $AG(R)$ that is not an edge of $\Gamma(R)$ for some distinct $x, y \in Z(R)^*$, then there is no path in $AG(R)$ of length two from x to y;*
4. *There are some distinct $x, y \in Z(R)^*$ such that $x - y$ is an edge of $AG(R)$ that is not an edge of $\Gamma(R)$ and there is no path in $AG(R)$ of length two from x to y;*

(5) *R is ring-isomorphic to either $\mathbb{Z}_2 \times \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^2)$.*

In view of Theorem 2.7, the following result follows

Corollary 2.8. *([23, Corollary 2.10]) Let R be a commutative ring such that $AG(R) \neq \Gamma(R)$ and assume that R is not ring-isomorphic to $\mathbb{Z}_2 \times B$, where $B = \mathbb{Z}_4$ or $B = \mathbb{Z}_2[X]/(X^2)$. If E is an edge of $AG(R)$ that is not an edge of $\Gamma(R)$, then E is an edge of a cycle of length three in $AG(R)$.*

A direct implication of Theorem 2.7 and Corollary 2.8 is the following result.

Corollary 2.9. *([23, Corollary 2.11]) Let R be a commutative ring such that $AG(R) \neq \Gamma(R)$. Then $gr(AG(R)) \in \{3, 4\}$.*

Theorem 2.10. *([24, Theorem 2.5]) Let R be a non-reduced ring such that R is not ring-isomorphic to $Z_2 \times B$, where $B = Z_4$ or $B = \frac{Z_2[X]}{(X^2)}$. Then the following statements are equivalent:*

1. *$gr(AG(R)) = \infty$;*
2. *$AG(R)$ is a star graph;*
3. *$AG(R)$ is a bipartite graph;*
4. *$AG(R)$ is a complete bipartite graph;*
5. *$\sum^* = Ass(R) = \{ann_R(x), ann_R(y)\}$ for some $x, y \in Z(R)^*$. Furthermore, if $ann_R(x) = ann_R(y)$, then $|ann_R(x)| = |Z(R)| = 3$ and if $ann_R(x) \neq ann_R(y)$, then $\sum^* = \{Z(R), ann_R(Z(R))\}$ and $|ann_R(Z(R))^*| = 1$.*

Theorem 2.11. *([24, Corollary 2.3]) Let R be a ring. Then $AG(R)$ is a complete bipartite graph if and only if one of the following statements holds:*

1. *$Nil(R) = \{0\}$ and $|Min(R)| = 2$;*
2. *$Nil(R) \neq \{0\}$ and either $AG(R) = K^{1,n}$ or $AG(R) = K^{2,3}$, where $1 \leq n \leq \infty$.*

Let x be a vertex of $AG(R)$. In the following result, the authors in[24] gave conditions under which x is adjacent to every vertex in $\Gamma(R)$.

Theorem 2.12. *([24, Theorem 2.6]) Let R be a ring and x be a vertex of $AG(R)$. Then the following statements are equivalent:*

1. *x is adjacent to every other vertex of $\Gamma(R)$;*
2. *$ann_R(x)$ is a maximal element of $\sum$ and x is adjacent to every other vertex of $AG(R)$.*

Recall that a undirected simple graph $G = (V, E)$ is called an n-partite graph if $V = A_1 \cup A_2 \cup \cdots \cup A_n$ for some $n \geq 2$, where each $A_i \neq \phi$, $A_i \cap A_j = \phi$, $i \neq j$, $1 \leq i, j \leq n$, and $x, y \in A_i$ implies $x - y$ is not an edge of G.

The authors in [25] prove the following result.

Theorem 2.13. *([25, Theorem 2.1]) Let $R = D_1 \times \cdots \times D_n$, where $n \geq 2$ and D_i is an integral domain for every $1 \leq i \leq n$. Then the following statements hold:*

1. *$AG(R)$ is an $nC\lceil \frac{n}{2} \rceil$-partite graph (Recall that mCn (m choose n) $= \frac{m!}{n!(m-n)!}$.)*
2. *$AG(R)$ is not an $nC\lceil \frac{n}{2} \rceil - 1$-partite graph.*

3. When does $AG(R) = \Gamma(R)$?

It is natural to ask when does $AG(R) = \Gamma(R)$? For a reduced ring R that is not an integral domain, we have the following results.

3.1. **Case I:** R **is reduced.**

Theorem 3.1. *(*[23, Theorem 3.3]*) Let R be a reduced commutative ring that is not an integral domain. Then the following statements are equivalent:*
(1) $AG(R)$ is complete;
(2) $\Gamma(R)$ is complete (and hence $AG(R) = \Gamma(R)$);
(3) R is ring-isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Theorem 3.2. *(*[23, Theorem 3.4]*) Let R be a reduced commutative ring that is not an integral domain and assume that $Z(R)$ is an ideal of R. Then $AG(R) \neq \Gamma(R)$ and $gr(AG(R)) = 3$.*

Theorem 3.3. *(*[23, Theorem 3.5]*) Let R be a reduced commutative ring with $|Min(R)| \geq 3$ (possibly $Min(R)$ is infinite). Then $AG(R) \neq \Gamma(R)$ and $gr(AG(R)) = 3$.*

Theorem 3.4. *(*[23, Theorem 3.6]*) Let R be a reduced commutative ring that is not an integral domain. Then $AG(R) = \Gamma(R)$ if and only if $|Min(R)| = 2$.*

Theorem 3.5. *(*[23, Theorem 3.7]*) Let R be a reduced commutative ring. Then the following statements are equivalent:*
(1) $gr(AG(R)) = 4$;
(2) $AG(R) = \Gamma(R)$ and $gr(\Gamma(R)) = 4$;
(3) $gr(\Gamma(R)) = 4$;
(4) $T(R)$ is ring-isomorphic to $K_1 \times K_2$, where each K_i is a field with $|K_i| \geq 3$;
(5) $|Min(R)| = 2$ and each minimal prime ideal of R has at least three distinct elements;
(6) $\Gamma(R) = K^{m,n}$ with $m, n \geq 2$;
(7) $AG(R) = K^{m,n}$ with $m, n \geq 2$.

Theorem 3.6. *(*[23, Theorem 3.8]*) Let R be a reduced commutative ring that is not an integral domain. Then the following statements are equivalent:*
(1) $gr(AG(R)) = \infty$;
(2) $AG(R) = \Gamma(R)$ and $gr(AG(R)) = \infty$;
(3) $gr(\Gamma(R)) = \infty$;
(4) $T(R)$ is ring-isomorphic to $Z_2 \times K$, where K is a field;
(5) $|Min(R)| = 2$ and at least one minimal prime ideal ideal of R has exactly two distinct elements;
(6) $\Gamma(R) = K^{1,n}$ for some $n \geq 1$;
(7) $AG(R) = K^{1,n}$ for some $n \geq 1$.

In view of Theorem 3.5 and Theorem 3.6, we have the following result.

Corollary 3.7. *(*[23, Corollary 3.9]*) Let R be a reduced commutative ring. Then $AG(R) = \Gamma(R)$ if and only if $gr(AG(R)) = gr(\Gamma(R)) \in \{4, \infty\}$.*

If R is non-reduced, then we have the following results.

3.2. **Case II:** R **is non-reduced.**

Theorem 3.8. *(*[24, Theorem 2.2]*) Let R be a ring such that for each edge of $AG(R)$, say $x - y$, either $ann_R(x) \in Ass(R)$ or $ann_R(y) \in Ass(R)$. Then $AG(R) = \Gamma(R)$. In particular, if $\sum^* = Ass(R)$, then $\Gamma(R) = AG(R)$.*

Theorem 3.9. *([24, Theorem 2.3]) Let R be a non-reduced ring. Then the following statements are equivalent:*

(1) $\Gamma(R) = AG(R) = K^n \vee \overline{K^m}$, *where* $n = |Nil(R)^*|$ *and* $m = |Z(R) \setminus Nil(R)|$.
(2) $ann_R(Z(R))$ *is a prime ideal of* R.
(3) $\sum^* = Ass(R)$ *and* $|\sum^*| \leq 2$.

Theorem 3.10. *([23, Theorem 3.15]) Let R be a non-reduced commutative ring such that $Z(R)$ is not an ideal of R. Then $AG(R) \neq \Gamma(R)$.*

Theorem 3.11. *([23, Theorem 3.16]) Let R be a non-reduced commutative ring. Then the following statements are equivalent:*
(1) $gr(AG(R)) = 4$;
(2) $AG(R) \neq \Gamma(R)$ *and* $gr(AG(R)) = 4$;
(3) R is ring-isomorphic to either $\mathbb{Z}_2 \times \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^2)$;
(4) $\Gamma(R) = \overline{K}^{1,3}$;
(5) $AG(R) = K^{2,3}$.

We observe that $gr(\Gamma(\mathbb{Z}_8)) = \infty$, but $gr(AG(\mathbb{Z}_8)) = 3$. We have the following result.

Theorem 3.12. *([23, Theorem 3.17]) Let R be a commutative ring such that $AG(R) \neq \Gamma(R)$. Then the following statements are equivalent:*
(1) $\Gamma(R)$ is a star graph;
(2) $\Gamma(R) = K^{1,2}$;
(3) $AG(R) = K^3$.

Theorem 3.13. *([23, Theorem 3.18]) Let R be a non-reduced commutative ring with $|Z(R)^*| \geq 2$. Then the following statements are equivalent:*
(1) $AG(R)$ is a star graph;
(2) $gr(AG(R)) = \infty$;
(3) $AG(R) = \Gamma(R)$ *and* $gr(\Gamma(R)) = \infty$;
(4) $Nil(R)$ is a prime ideal of R and either $Z(R) = Nil(R) = \{0, -w, w\}$ $(w \neq -w)$ for some nonzero $w \in R$ or $Z(R) \neq Nil(R)$ and $Nil(R) = \{0, w\}$ for some nonzero $w \in R$ (and hence $wZ(R) = \{0\}$);
(5) Either $AG(R) = K^{1,1}$ or $AG(R) = K^{1,\infty}$;
(6) Either $\Gamma(R) = K^{1,1}$ or $\Gamma(R) = K^{1,\infty}$.

Corollary 3.14. *([23, Corollary 3.19]) Let R be a non-reduced commutative ring with $|Z(R)^*| \geq 2$. Then $\Gamma(R)$ is a star graph if and only if $\Gamma(R) = K^{1,1}$, $\Gamma(R) = K^{1,2}$, or $\Gamma(R) = K^{1,\infty}$.*

Remark 3.15. *In view of Theorem 2.10, the authors in [24] gave an alternative proof of Theorem 3.13 (see [24, Corollary 2.4]).*

In the following example, we construct two non-reduced commutative rings say R_1 and R_2, where $AG(R_1) = K^{1,1}$ and $AG(R_2) = K^{1,\infty}$.

Example 3.16.
(1) *Let $R_1 = \mathbb{Z}_3[X]/(X^2)$ and let $x = X + (X^2) \in R_1$. Then $Z(R_1) = Nil(R_1) = \{0, -x, x\}$ and $AG(R_1) = \Gamma(R_1) = K^{1,1}$. Also note that $AG(\mathbb{Z}_9) = \Gamma(\mathbb{Z}_9) = K^{1,1}$.*
(2) *Let $R_2 = \mathbb{Z}_2[X, Y]/(XY, X^2)$. Then let $x = X + (XY + X^2)$ and $y = Y + (XY + X^2) \in R_2$. Then $Z(R_2) = (x, y)R_2$, $Nil(R_2) = \{0, x\}$, and $Z(R_2) \neq Nil(R_2)$. It is clear that $AG(R_2) = \Gamma(R_2) = K^{1,\infty}$.*

Remark 3.17. *Let R be a non-reduced commutative ring. In view of Theorem 3.10, Theorem 3.11, and Theorem 3.13, if $AG(R) = \Gamma(R)$, then $Z(R)$ is an ideal of R and $gr(AG(R)) = gr(\Gamma(R)) \in \{3, \infty\}$. The converse is true if $gr(AG(R) = gr(\Gamma(R)) = \infty$ (see Theorem 3.10 and 3.13). However, if $Z(R)$ is an ideal of R and $gr(AG(R)) = gr(\Gamma(R)) = 3$, then it is possible to have all the following cases:*

1. *It is possible to have a commutative ring R such that $Z(R)$ is an ideal of R, $Z(R) \neq Nil(R)$, $AG(R) = \Gamma(R)$, and $gr(AG(R)) = 3$. See Example 3.18.*
2. *It is possible to have a commutative ring R such that $Z(R)$ is an ideal of R, $Z(R) \neq Nil(R)$, $Nil(R)^2 = \{0\}$, $AG(R) \neq \Gamma(R)$, $diam(AG(R)) = diam(\Gamma(R)) = 2$, and $gr(AG(R)) = gr(\Gamma(R)) = 3$. See Example 3.19.*
3. *It is possible to have a commutative ring R such that $Z(R)$ is an ideal of R, $Z(R) \neq Nil(R)$, $Nil(R)^2 = \{0\}$, $AG(R)$ is a complete graph (i.e., $diam(AG(R)) = 1$), $AG(R) \neq \Gamma(R)$, $diam(\Gamma(R)) = 2$, and $gr(AG(R)) = gr(\Gamma(R)) = 3$. See Theorem 3.20.*

Example 3.18. *Let $D = \mathbb{Z}_2[X, Y, W]$, $I = (X^2, Y^2, XY, XW)D$ is an ideal of D, and let $R = D/I$. Then let $x = X + I, y = Y + I$, and $w = W + I$ be elements of R. Then $Nil(R) = (x, y)R$ and $Z(R) = (x, y, w)R$ is an ideal of R. By construction, we have $Nil(R)^2 = \{0\}$, $AG(R) = \Gamma(R)$, $diam(AG(R)) = diam(\Gamma(R)) = 2$, and $gr(AG(R)) = gr(\Gamma(R)) = 3$ (for example, $x - (x + y) - y - x$ is a cycle of length three).*

Example 3.19. *Let $D = \mathbb{Z}_2[X, Y, W]$, $I = (X^2, Y^2, XY, XW, YW^3)D$ is an ideal of D, and let $R = D/I$. Then let $x = X + I, y = Y + I$, and $w = W + I$ be elements of R. Then $Nil(R) = (x, y)R$ and $Z(R) = (x, y, w)R$ is an ideal of R. By construction, $Nil(R)^2 = \{0\}$, $diam(AG(R)) = diam(\Gamma(R)) = 2$, $gr(AG(R)) = gr(\Gamma(R)) = 3$. However, since $w^3 \neq 0$ and $y \in ann_R(w^3) \setminus (ann_R(w) \cup ann_R(w^2))$, we have $w - w^2$ is an edge of $AG(R)$ that is not an edge of $\Gamma(R)$, and hence $AG(R) \neq \Gamma(R)$.*

Given a commutative ring R and an R-module M, the *idealization* of M is the ring $R(+)M = R \times M$ with addition defined by $(r, m) + (s, n) = (r + s, m + n)$ and multiplication defined by $(r, m)(s, n) = (rs, rn + sm)$ for all $r, s \in R$ and $m, n \in M$. Note that $\{0\}(+)M \subseteq Nil(R(+)M)$ since $(\{0\}(+)M)^2 = \{(0, 0)\}$. We have the following result

Theorem 3.20. *(*[23, Theorem 3.24]*) Let D be a principal ideal domain that is not a field with quotient field K (for example, let $D = \mathbb{Z}$ or $D = F[X]$ for some field F) and let $Q = (p)$ be a nonzero prime ideal of D for some prime (irreducible) element $p \in D$. Set $M = K/D_Q$ and $R = D(+)M$. Then $Z(R) \neq Nil(R)$, $AG(R)$ is a complete graph, $AG(R) \neq \Gamma(R)$, and $gr(AG(R)) = gr(\Gamma(R)) = 3$.*

The following example shows that the hypothesis "Q is principal" in the above Theorem is crucial.

Example 3.21. *Let $D = \mathbb{Z}[X]$ with quotient field K and $Q = (2, X)D$. Then Q is a nonprincipal prime ideal of D. Set $M = K/D_Q$ and $R = D(+)M$. Then $Z(R) = Q(+)M$, $Nil(R) = \{0\}(+)M$, and $Nil(R)^2 = \{(0, 0)\}$. Let $a = (2, 0)$ and $b = (0, \frac{1}{X} + D_Q)$. Then $ab = (0, \frac{2}{X} + D_Q) \in Nil(R)^*$. Since $ann_R(ab) = ann_R(b)$, we have $a - b$ is not an edge of $AG(R)$. Thus $AG(R)$ is not a complete graph.*

We terminate this section with the following open question.

(Open question, [24]**)**: Let R be a non-reduced ring and $x-y$ be an edge of $AG(R)$. If $\Gamma(R) = AG(R)$, then is it true either $ann_R(x) \in Ass(R)$ or $ann_R(y) \in Ass(R)$?

4. Clique number and chromatic number of $AG(R)$

Let $G = (V, E)$ be a graph. The clique number of G, denoted by $w(G)$, is the largest positive integer n such that K_n is a subgraph of G. The chromatic number of G, denoted by $\chi(G)$, is the the minimal number of colors which can be assigned to the vertices of G in such a way that every two adjacent vertices have different colors. It should be clear that $w(G) \leq \chi(G)$. Again, recall that mCn (m choose n) $= \frac{m!}{n!(m-n)!}$.

Theorem 4.1. *(*[25, Theorem 2.2]*) Assume that R is ring-isomorphic to $D_1 \times \cdots \times D_n$, where $n \geq 2$ and D_i is an integral domain for every $1 \leq i \leq n$. Then $w(AG(R)) = \chi(AG(R)) = nC\lceil \frac{n}{2} \rceil$. In particular, if R is an Artinian ring, then $w(AG(R)) = \chi(AG(R)) = |Max(R)|C\lceil \frac{|Max(R)|}{2} \rceil$.*

Theorem 4.2. *(*[25, Theorem 2.3]*) Let R be a non-reduced ring. Then the following statements hold.*

1. *If $|Z(R)| < \infty$, then the following statements are equivalent:*
   (a) *$w(AG(R)) = |Nil(R)|$.*
   (b) *$\chi(AG(R)) = |Nil(R)|$.*
   (c) *$AG(R) = K^{2,3}$.*
2. *If $|Z(R)| = \infty$, $w(AG(R)) < \infty$ and $Z(R)$ is an ideal of R, then the following statements are equivalent:*
   (a) *$W(AG(R)) = |Nil(R)$.*
   (b) *$\chi(AG(R)) = |Nil(R)|$.*
   (c) *$AG(R) = K_{|Nil(R)^*|} \vee \overline{K_\infty}$.*
   (d) *$x-y$ is not an edge of $AG(R)$, for every $x, y \in Z(R) \setminus Nil(R)$.*

It is well-known that if G is a bipartite graph, then $\chi(AG(R)) = 2$. In the following result, the authors in [25] classified all bipartite annihilator graphs of rings.

Theorem 4.3. *(*[25, Theorem 2.4]*) Let R be a non-reduced ring. Then the following statements are equivalent:*

1. *$w(AG(R)) = 2$;*
2. *$\chi(AG(R)) = 2$;*
3. *$AG(R) = K_{2,3}$ or $AG(R) = K_2$ or $AG(R) = K_1 \vee \overline{K_\infty}$.*

5. Genus of $AG(R)$

The genus of a graph G, denoted by $g(G)$, is the minimal integer n such that the graph can be embedded in S_n. Intuitively, G is embedded in a surface if it can be drawn in the surface so that its edges intersect only at their common vertices. A graph G with genus 0 is called a planar graph and a graph G with genus 1 is called as a toroidal graph. Note that if H is a subgraph of a graph G, then $g(H) \leq g(G)$. In the following result, the authors in [31] classified all quasi-local rings (up to isomorphism) that have planar annihilator graphs.

Theorem 5.1. *([31, Theorem 15]) Let R be a quasi-local ring. Then $AG(R)$ is a planar if and only if R is ring-isomorphic to one of the following rings:* Z_4, $\frac{Z_2[X]}{(X^2)}$, Z_9, $\frac{Z_3[X]}{(X^3)}$, Z_8, $\frac{Z_2[X]}{(X^3)}$, $\frac{Z_4[X]}{(X^3,X^2-2)}$, $\frac{Z_2[X,Y]}{(X^2,XY,Y^2)}$, $\frac{Z_4[X]}{(2X,X^2)}$, $\frac{F_4[X]}{(X^2)}$ *(where F_4 denotes a field with 4 elements),* $\frac{Z_4[X]}{(X^2+X+1)}$, Z_{25}, *or* $\frac{Z_5[X]}{(X^2)}$.

For a reduced finite ring, we have the following result.

Theorem 5.2. *([31, Theorem 16]) Let R be a reduced finite ring that is not a field, i.e., $R = F_1 \times \cdots \times F_n$, where each F_i is a finite field and $n \geq 2$. Then $AG(R)$ is planar if and only if R is ring-isomorphic to one of the following rings: $Z_2 \times F, Z_3 \times F, Z_2 \times Z_2 \times Z_2, Z_2 \times Z_2 \times Z_3$, where F is a finite field.*

If R is a non-reduced finite ring, then we have the following.

Theorem 5.3. *([31, Theorem 17]) Assume that R is ring-isomorphic to $R_1 \times \cdots \times R_n \times F_1 \times \cdots \times F_m$, where each R_i is a finite quasi-local ring that is not a field, each F_i is a finite field, and $n, m \geq 1$. Then $AG(R)$ is planar if and only if R is ring-isomorphic to one of the following rings: $Z_4 \times Z_2$, $\frac{Z_2[X]}{(X^2)} \times Z_2$.*

The following result classifies (up to isomorphism) all quasi-local rings that have genus one annihilator graphs.

Theorem 5.4. *([31, Theorem 18]) Let R be a quasi-local ring. Then $g(AG(R)) = 1$ if and only if R is ring-isomorphic to one of the following rings:*
Z_{16}, $\frac{Z_2[X]}{(X^4)}$, $\frac{Z_4[X]}{(X^4,X^2-2)}$, $\frac{Z_2[X]}{(X^3-2,X^4)}$, $\frac{Z_4[X]}{(X^3+X^2-2,X^4)}$, $\frac{Z_2[X]}{(X^3,X^2-2X)}$, $\frac{Z_2[X,Y]}{(X^3,XY,Y^2-X^2)}$, $\frac{Z_8[X]}{(X^2-4,2x)}$, $\frac{Z_4[X,Y]}{(X^3,XY,X^2-2,Y^2-2,Y^3)}$, $\frac{Z_4[X]}{(X^2)}$, $\frac{Z_4[X,y]}{(X^2,Y^2,XY-2))}$, $\frac{Z_2[X,Y]}{(X^2,Y^2)}$, $\frac{Z_2[X,Y]}{(X^2,Y^2,XY)}$, $\frac{Z_4[X]}{(X^3,2x)}$, $\frac{Z_4[X,Y]}{(X^3,X^2-2,XY,Y^2)}$, $\frac{Z_8[X]}{(X^2)}$, $\frac{F_8[X]}{(X^2)}$, $\frac{Z_4[X]}{(X^3+X+1)}$, $\frac{Z_4[X,Y]}{(2X,2Y,X^2,Y^2,XY)}$, $\frac{Z_2[X,Y,Z]}{(X,Y,Z)^2}$, Z_{49}, *or* $\frac{Z_7[X]}{(X^2)}$

The following result classifies (up to isomorphism) all finite reduced rings that have genus one annihilator graphs.

Theorem 5.5. *([31, Theorem 19]) Let R be a reduced finite ring that is not a field, i.e.,R is ring-isomorphic to $F_1 \times \cdots \times F_n$, where each F_i is a finite field and $n \geq 2$. Then $g(AG(R)) = 1$ if and only if R is ring-isomorphic to one of the following rings: $F_4 \times F_4, F_4 \times Z_5, Z_5 \times Z_5$, or $F_4 \times Z_7$.*

If R is a non-reduced finite ring, then we have the following.

Theorem 5.6. *([31, Theorem 20])Assume that R is ring-isomorphic to $R_1 \times \cdots \times R_n \times F_1 \times \cdots \times F_m$, where each R_i is a finite quasi-local ring that is not a field, each F_i is a finite field, and $n, m \geq 1$. Then $g(AG(R)) = 1$ if and only if R is ring-isomorphic to one of the following rings: $Z_4 \times Z_3$, or $\frac{Z_2[X]}{(X^2)} \times Z_3$.*

6. Extended zero-divisor graph of R: $EG(R)$

Recall ([26]) that the extended zero-divisor graph of R is the undirected (simple) graph $EG(R)$ with the vertex set $Z(R)^*$, and two distinct vertices x and y are adjacent if and only if either $Rx \cap ann_R(y) \neq \{0\}$ or $Ry \cap ann_R(x) \neq \{0\}$. Hence it follows that the zero-divisor graph $\Gamma(R)$ is a subgraph of $EG(R)$.

In the following result, we collect some basic properties of $EG(R)$.

Theorem 6.1. *([26]) Let R be a ring. Then*

(1) *([26, Theorem 2.1]) $EG(R)$ is connected and $diam(EG(R)) \leq 2$. Moreover, if $E(G)$ has a cycle, then $gr(EG(R)) \leq 4$.*
(2) *([26, Theorem 2.2]) If $EG(R)$ has a cycle, then $gr(E(G)) = 4$ if and only if R is reduced with $|Min(R)| = 2$.*
(3) *([26, Theorem 3.2]) $EG(R)$ is a star graph if and only if one of the following statements holds:*
   (a) *R is ring-isomorphic to $Z_2 \times D$, where D is an integral domain.*
   (b) *$|Z(R)| = 3$.*
   (c) *$Nil(R)$ is a prime ideal of R and $|Nil(R)| = 2$.*
(4) *([26, Theorem 3.3]) Suppose that R is a non-reduced ring such that $EG(R)$ is a star graph. Then the following statements hold:*
   (a) *R is indecomposable.*
   (b) *Either $|Z(R)| = 3$ or $|Z(R)| = \infty$.*
(5) *Assume that R is ring-isomorphic to $D_1 \times \cdots \times D_n$, where $n \geq 2$ and each D_i is an integral domain. Then $EG(R)$ is a complete $(2^n - 2)$-partite graph.*

Let R be a ring and $x, y \in R$. The authors in [26] called an element x an Ry-regular element if $x \notin Z(Ry)$ and $RxRy \neq Ry$.

Theorem 6.2. *([26, Theorem 3.5]) Let R be a non-reduced ring. Then $EG(R)$ is complete if and only if R is indecomposable and either x is not Ry-regular or y is not Rx-regular, for every distinct $x, y \in Z(R)^*$.*

7. When does $EG(R) = \Gamma(R)$?

Since $\Gamma(R)$ is always an induced subgraph of $EG(R)$, it is natural to ask when does $EG(R) = \Gamma(R)$? First, we consider the case when R is reduced.

7.1. **Case I: R is reduced.**

Theorem 7.1. *([26]) Let R be a reduced ring that is not an integral domain.*
(1) *([26, Theorem 4.1, Corollary 4.3]) Assume $|Min(R)| = n$. The following statements are equivalent:*
   (a) *$n = 2$;*
   (b) *$\Gamma(R) = EG(R)$;*
   (c) *$gr(EG(R)) = gr(\Gamma(R)) \in \{4, \infty\}$.*
(2) *([26, Corollary 4.1]) The following statements are equivalent:*
   (a) *$gr(EG) = \infty$;*
   (b) *$EG(R) = \Gamma(R)$ and $gr(EG(R)) = \infty$;*
   (c) *$gr(\Gamma(R)) = \infty$;*
   (d) *$|Min(R)| = 2$ and at least one minimal prime ideal of R has exactly two distinct elements;*
   (e) *$\Gamma(R) = K_{1,n}$ for some $n \geq 1$.*
   (f) *$EG(R) = K_{1,n}$ for some $n \geq 1$.*
(3) *([26, Corollary 4.2]) The following statements are equivalent:*
   (a) *$gr(EG(R)) = 4$;*
   (b) *$EG(R) = \Gamma(R)$ and $gr(\Gamma(R)) = 4$;*
   (c) *$gr(EG(R)) = 4$;*
   (d) *$|Min(R)| = 2$ and each minimal prime ideal of R has at least three distinct elements;*
   (e) *$EG(R) = K_{m,n}$ for some $m, n \geq 2$;*

(f) $\Gamma(R) = K_{m,n}$ *for some* $m, n \geq 2$.

Now we consider the case when R is non-reduced.

7.2. **Case II: R is non-reduced.**

Theorem 7.2. *(*[26, Theorem 4.3]*) Let R be a non-reduced ring. Then the following statements are equivalent:*

(1) $gr(EG(R)) = \infty$;
(2) $EG(R)$ *is a star graph;*
(3) $EG(R) = \Gamma(R)$ *and* $gr(\Gamma(R)) = \infty$;
(4) $ann_R(Z(R))$ *is a prime ideal of R and either* $|Z(R)| = |ann_R(Z(R))| = 3$ *or* $|ann_R(Z(R))| = 2$ *and* $|Z(R)| = \infty$;
(5) $EG(R) = K_{1,1}$ *or* $EG(R) = K_{1,\infty}$;
(6) $\Gamma(R) = K_{1,1}$ *or* $\Gamma(R) = K_{1,\infty}$.

8. When is $EG(R)$ planar?

Recall that a graph G is called a planar if it can be drawn in the plane so that the edges of G do not cross.

Theorem 8.1. *(*[27, Theorem 3.2]*) Let R be a ring such that either R is ring-isomorphic to $R_1 \times R_2 \times R_3$ (for some rings R_1, R_2, R_3) or $|Min(R)| \geq 3$ and R is ring-isomorphic to $R_1 \times R_2$(for some rings R_1, R_2), then $EG(R)$ is not a planar.*

For a reduced ring R, we have the following result.

Theorem 8.2. *(*[27, Theorem 3.3]*) Let R be a reduced ring. Then the following statements hold:*

(1) $EG(R)$ *is planar;*
(2) $|Min(R)| = 2$ *and one of the minimal prime ideals of R has at most three distinct elements.*

For a non-reduced ring R, we have the following result.

Theorem 8.3. *(*[27]*) Let R be a non-reduced ring. Then*

(1) *(*[27, Theorem 3.4]*) Suppose that R is not ring-isomorphic to either Z_4 or $\frac{Z_2[X]}{(X^2)}$. Then*
   (a) *Suppose that $|Z(R)| < \infty$. Then $EG(R)$ is planar if and only if R is ring-isomorphic to either $Z_2 \times Z_4$ or $Z_2 \times \frac{Z_2[X]}{(X^2)}$.*
   (b) *Suppose that $|Z(R)| = \infty$. Then $EG(R)$ is planar if and only if $ann_R(R)$ is a prime ideal of R.*
(2) *(*[27, Theorem 3.5]*) Suppose that $|Nil(R)| = 3$. Then $ann_R(Z(R))$ is a prime ideal of R if and only if $EG(R)$ is planar.*
(3) *(*[27, Theorem 3.6]*) If $|Nil(R)| \geq 6$, then $EG(R)$ is not planar. If $4 \leq |Nil(R)| \leq 5$, then $EG(R)$ is planar if and only if $Z(R) = Nil(R)$.*

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